FLOW OF A LIQUID IN A TUBE WITH GRID ELECTRODES IN THE PRESENCE OF WEAK MAGNETOHYDRODYNAMIC INTERACTION

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The liquid is inviscid, incompressible, and conducting; the tube is of circular cross-section, with insulating walls. The two electrodes are at right angles to the axis and a distance l apart (Fig. 1), and have constant potentials.

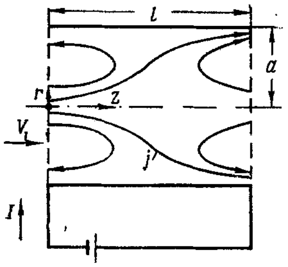

Fig. 1

The equations of magnetohydrodynamics then take the form

$$\operatorname{div}\mathbf{V}=0,\quad (\mathbf{V}\nabla)\mathbf{V}=-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\mathbf{j}\times\mu\mathbf{H},$$

$$\mathbf{j}=\operatorname{rot}\mathbf{H},\quad \mathbf{j}=\sigma(\mathbf{E}+\mathbf{V}\times\mu\mathbf{H}). \qquad (1)$$

The conditions for axial symmetry

$$\frac{\partial}{\partial\nu}=0,\quad \mathbf{H}=(0,H_\vartheta,0),\qquad \mathbf{V}=(V_r,0,V_z)$$

and the electric potential φ ($E=-\nabla\varphi$) give us five equations for V_r, V_z, H, p, and φ:

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{\partial V_z}{\partial z}=0,$$

$$V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\mu}{\rho}H\frac{1}{r}\frac{\partial}{\partial r}(Hr),$$

$$V_r\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-$$

$$-\frac{\mu}{\rho}H\frac{\partial H}{\partial z}=-\frac{1}{\rho}\frac{\partial}{\partial z}\left(p+\mu\frac{H^2}{2}\right),$$

$$-\frac{\partial H}{\partial z}=\sigma\left(-\frac{\partial\varphi}{\partial r}-\mu HV_z\right)=j_r,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(Hr)=\sigma\left(-\frac{\partial\varphi}{\partial z}-\mu HV_r\right)=j_z. \qquad (2)$$

We introduce the following dimensionless quantities (in which V_1 is speed of liquid and I is current):

$$z^\circ=\frac{z}{a},\quad r^\circ=\frac{r}{a},\quad V^\circ=\frac{V}{V_1},$$

$$H^\circ=\frac{H}{H_a},\quad H_a=\frac{I}{2\pi a},$$

$$R_m=\mu\sigma V_1 a,\qquad A^2=\frac{\mu H_a^2}{\rho V_1^2}.$$

Then the equations of (2) become, with the superscript zero omitted,

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{\partial V_z}{\partial z}=0,$$

$$V_r\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=-\frac{\partial p}{\partial z}-A^2H\frac{\partial H}{\partial z},$$

$$V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}=-\frac{\partial p}{\partial r}-A^2H\frac{1}{r}\frac{\partial}{\partial r}(rH),$$

$$\frac{\partial H}{\partial z}=R_m\left(\frac{\partial\varphi}{\partial r}+V_zH\right),$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH)=R_m\left(-\frac{\partial\varphi}{\partial z}+V_rH\right). \qquad (3)$$

We envisage the particular case $A^2 \ll 1$, i.e., the magnetic pressure is much less than the dynamic pressure; then we may neglect the latter terms in the second and third equations of (3).

As regards R_m, we merely assume that $R_m \ll 1/A^2$, so R_m may be large if A^2 is sufficiently small.

It is therefore assumed that the interaction parameter A^2R_m is very small, and so the ponderomotive forces do not affect the flow, and the solution for the velocities becomes $V_z=\pm1$, $V_r=0$, i.e., the liquid moves as a solid rod along its axis.

We eliminate φ from the last two equations of (3) to get a second-order differential equation for the magnetic field:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH)\right)+\frac{\partial^2H}{\partial z^2}\right]=V_zR_m\frac{\partial H}{\partial z}. \qquad (4)$$

We then have to find a solution to Eq. (4) satisfying the boundary conditions

$$H=1\qquad \text{for } r=1,\ 0\leqslant z\leqslant l,$$

$$\frac{\partial H}{\partial z}=V_zR_mH\quad\text{for } z=0,\qquad \frac{\partial H}{\partial z}=V_zR_mH\quad\text{for } z=l,$$

$$H=0\quad\text{for } r=0. \qquad (5)$$

This solution is of the form

$$H=2V_zR_m\times$$

$$\times\sum_{k=1}^{\infty}\frac{(e^{-\beta l}-1)\,\alpha e^{\alpha z}+(e^{\alpha l}-1)\,\beta e^{-\beta z}}{(e^{\alpha l}-e^{-\beta l})\,\lambda_k^3J_0(\lambda_k)}J_1(\lambda_k r)+r. \qquad (6)$$

Here J_0 is a Bessel function of zero order, J_1 is a Bessel function of first order, and λ_k is a root of $J_1(\lambda)=0$;

$$\alpha={}^1/_2V_zR_m+\sqrt{{}^1/_4R_m^2+\lambda_k^2},$$

$$\beta=-{}^1/_2V_zR_m+\sqrt{{}^1/_4R_m^2+\lambda_k^2}. \qquad (7)$$

Figure 2 shows the field distribution within the tube calculated from Eq. (6), with curves 1–3 for z of 0, $l/2$, and l respectively. The magnetic field has a maximum around one electrode and may be greater than the field produced at the surface of the tube by the current in the external circuit, while the field is minimal around the other electrode. This shows that there are closed currents within the liquid that do not appear in the external circuit.

Figure 1 shows the current distribution deduced from the field pattern (Fig. 2), which applies for R_m sufficiently large; if R_m is small, we get a trivial distribution, namely straight-line flow from one electrode to the other.

The dimensionless potential difference at r = 1 is

$$\Delta\varphi(z)=-\int_0^z\frac{1}{R_m}\frac{1}{r}\frac{\partial}{\partial r}(rH)\,dz=$$

$$=2V_z\sum_{k=1}^{\infty}\frac{(e^{-\beta l}-1)(1-e^{\alpha z})-(e^{\alpha l}-1)(1-e^{-\beta z})}{(e^{\alpha l}-e^{\beta l})\,\lambda_k^2}-\frac{2z}{R_m}.$$

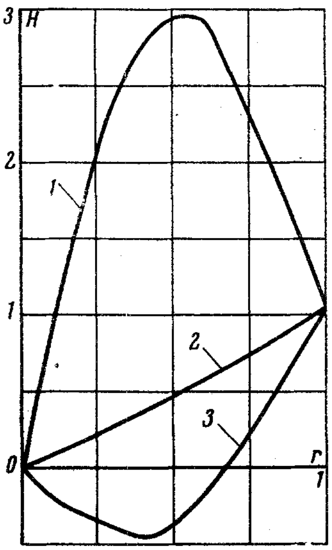

Fig. 2

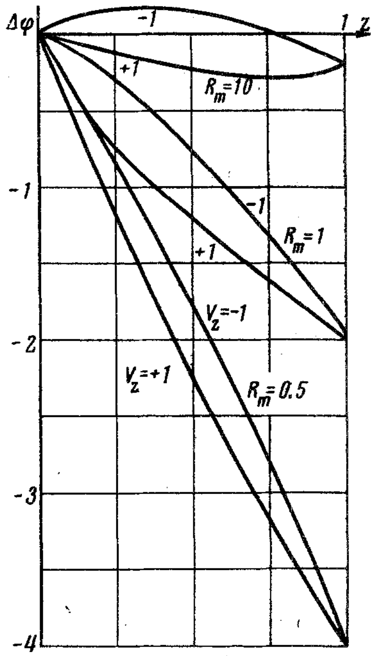

Fig. 3

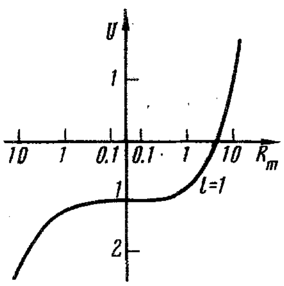

Fig. 4

If $z = l$,

$$\Delta\varphi = \varphi(l) - \varphi(0) = -\frac{2l}{R_m}.$$

Figure 3 shows $\Delta\varphi$ as a function of z for R_m of 0.5, 1, and 10, while Fig. 4 shows $U = \Delta\varphi R_m$ as a function of R_m for z = 0.5.

The potential difference between the electrodes is clearly directly proportional to channel length, inversely proportional to R_m, and always negative; $\Delta\varphi(z)$ may become positive within the channel for R_m sufficiently great.